ON CORRESPONDENCE OF BRST-BFV, DIRAC AND REFINED ALGEBRAIC QUANTIZATIONS OF CONSTRAINED SYSTEMS

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Abstract

Correspondence between BRST-BFV, Dirac and refined algebraic (group averaging, projection operator) approaches to quantize constrained systems is analyzed. For the closed-algebra case, it is shown that the component of the BFV wave function with maximal (minimal) number of ghosts and antighosts in the Schrodinger representation may be viewed as a wave function in the refined algebraic (Dirac) quantization approach. The Giulini-Marolf group averaging formula for the inner product in the refined algebraic quantization approach is obtained from the Batalin-Marnelius prescription for the BRST-BFV inner product which should be generally modified due to topological problems. The considered prescription for the correspondence of states is observed to be applicable to the open-algebra case. Refined algebraic quantization approach is generalized then to the case of nontrivial structure functions. A simple example is discussed. Correspondence of observables in different quantization methods is also investigated.

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I. INTRODUCTION

Theory of constrained systems is a basis of modern physics: gauge field theories, quantum gravity and supergravity, string and superstring models are examples of systems with constraints. For such theories, one should specify not only an evolution equation but additional requirements (constraints) imposing on initial conditions. Alternatively, one can take the constraints into account by modifying the inner product of the theory. For some cases, Hamiltonian is zero, so that all the dynamics is involved to constraints, and the well-known problem of time in reparametrization-invariant theories arises.

Different approaches have been developed for quantizing constrained systems. Each procedure of quantization has advantages and disadvantages. The most difficult problem is to introduce an inner product.

Quantization approaches can be divided into two classes: without introducing additional degrees of freedom and with introducing ghosts, antighosts, Lagrange multipliers and canonically conjugated momenta. The latter class of quantization methods is known as BRST-BFV approach [1,2,4] (for a review see [3]).

Examples of quantization techniques of the first class are Dirac method [5] (including gauge-fixing approach [5,6]) and refined algebraic quantization (group averaging method) [7,8] (analogous ideas are used in projection operator approach [9,10] and in lattice gauge theories [11]). The Dirac quantization is based on imposing the constraint conditions on physical states. It is not easy to introduce an inner product in the Dirac approach. However, the refined algebraic quantization method allows us to overcome this difficulty: instead of imposing constraints on physical states, one modifies the physical inner product due to constraints.

By now, refined algebraic quantization has been developed for the closed algebras of constraints only. It has been stressed in [12] that generalization of this method to the case of nontrivial structure functions is an interesting open problem.

The purposes of this paper are:

- to generalize the refined algebraic quantization approach to the case of constraint algebras with nontrivial structure functions;

- to investigate the correspondence between states and observables in Dirac, refined algebraic and BRST-BFV quantization approaches.

This paper is organized as follows. In section 2 the Dirac, refined algebraic and BRST-BFV quantization approaches are reviewed. General requirements for the refined algebraic inner product for the open-algebra case are formulated. Section 3 deals with finding the correspondence between states in BRST-BFV, Dirac and refined algebraic quantization approaches. The obtained formulas are justified for different known prescriptions of the BRST-BFV inner product for the closed-algebra case. This correspondence is supposed to be valid for the open-algebra case as well. This allows us to obtain a formula for the refined algebraic quantization inner product for the open-algebra case in section 4. The correspondence between observables in different quantization approaches is discussed in section 5. Section 6 deals with a simple example of a constrained system with nontrivial structure functions. Section 7 contains concluding remarks.

II. QUANTIZATION METHODS

A. Dirac approach

The most famous way to quantize a constrained system is the Dirac approach [5]. It is as follows. Evolution transformation is presented as $\exp(-iH^+t)$ for some Hamiltonian H. Usually, H is Hermitian, $H^+ = H$. However, the examples of non-Hermitian Hamiltonians are presented below. Constraints are taken into account by imposing the additional conditions:

$$\hat{\Lambda}_a^+ \Psi = 0, \qquad a = \overline{1, M}, \tag{2.1}$$

on the states. The operators $\hat{\Lambda}_a^+$ are quantum analogs of constraints. For the closed-algebra unimodular case, constraints are Hermitian, $\hat{\Lambda}_a^+ = \hat{\Lambda}_a$. Requirements (2.1) do not contradict each other, provided that the constraints commute on the constraint surface. For the quantum case, this means that

$$[\hat{\Lambda}_a^+, \hat{\Lambda}_b^+] = i(U_{ab}^c)^+ \hat{\Lambda}_c^+ \tag{2.2}$$

for some operators U_{ab}^c called usually as structure functions [3]. Relation (2.1) should also conserve under time evolution, so that constraints should commute with the Hamiltonian for the states obeying eq.(2.1):

$$[H^+, \hat{\Lambda}_a^+] = i(R_a^c)^+ \hat{\Lambda}_c^+ \quad (2.3)$$

for some operators R_a^c.

It is not easy to construct an inner product in the Dirac approach since $\Psi(q)$ are distributions rather than square-integrable functions.

Example 1. Consider the case $M = 1$, $q = x$, $\hat{\Lambda} = -i\frac{\partial}{\partial x}$. Then condition (2.1) will take the form $\frac{\partial \Psi}{\partial x} = 0$, so that $\Psi = const$ is not square integrable.

One usually imposes additional gauge conditions [5,3,6] in such a way that each gauge orbit should be taken into account once (for example 1, such gauge condition may be chosen as $x = 0$). The wave function $\Psi(q)$ is considered then on the gauge surface only, since the off-gauge-surface values Ψ are specified from the constraint equations (2.1); then the integral of $|\Psi|^2$ is taken over the gauge surface only. Unfortunately, this approach is gauge-dependent, especially for the case of the Gribov copies problem [13,10].

B. Refined algebraic quantization approach

An alternative way to develop the quantum theory is to use the refined algebraic approach [7,8] and take the constraints into account by modifying the inner product instead of imposing requirements (2.1). States are specified by smooth and damping at the infinity wave functions $\Phi(q)$ called auxiliary state vectors. However, their inner product is given by a nontrivial formula. Let us consider abelian and nonabelian cases.

1. Abelian case

For the simplest abelian case ($\hat{\Lambda}_a^+ = \hat{\Lambda}_a$, $U_{ab}^c = 0$) and constraints with continuous spectra, the inner product reads:

$$(\Phi, \prod_a 2\pi\delta(\hat{\Lambda}_a)\Phi). \quad (2.4)$$

Since it is degenerate, one should factorize the space of auxiliary states $\mathcal{H}_{aux}$: wave functions Φ_1 and Φ_2 are set to be equivalent if $(\Phi, \prod_a 2\pi\delta(\Lambda_a)(\Phi_1 - \Phi_2)) = 0$ for all Φ. The corresponding factorspace $\mathcal{H}_{aux}/\sim$ should be completed to obtain a physical Hilbert state space $\overline{\mathcal{H}_{aux}/\sim}$.

In particular, the following transformation

$$\Phi \to \Phi + \hat{\Lambda}_a X^a \quad (2.5)$$

takes a auxiliary state to the equivalent state and can be called as a quantum gauge transformation.

For example 1, formula (2.4) for the inner product takes the form $|\int_{-\infty}^{+\infty} dx\Phi(x)|^2$. Two auxiliary functions are then equivalent if their integrals $\int dx\Phi(x)$ coincide. The classes of equivalence are specified by numbers $\int dx\Phi(x)$, so that the physical Hilbert space is trivial.

Formulas of the Dirac approach are indeed reproduced in the refined algebraic quantization approach. For each auxiliary state, consider the wave function (distribution) [7]

$$\Psi = \prod_a 2\pi\delta(\hat{\Lambda}_a)\Phi. \quad (2.6)$$

For equivalent auxiliary states, we obtain the same Ψ. Thus, physical states can be specified by distributions Ψ obeying the Dirac condition (2.1). Moreover, we see that the inner product for the Dirac states is introduced. For Dirac wave functions Ψ_1 and Ψ_2 satisfying (2.1), one should find Φ_1 and Φ_2 from relation (2.6) and evaluate the quantity $(\Phi_1, \prod_a 2\pi\delta(\hat{\Lambda}_a)\Phi_2)$. This result will not depend on the particular choice of representatives Ψ_1 and Ψ_2 of the equivalence classes.

For example 1, formula (2.6) is taken to the form $\Psi(x) = \int dy\Phi(y)$. We see that Ψ is x-independent. Moreover, for constant functions Ψ_1 and Ψ_2 we find their inner product $\Psi_1^*\Psi_2$.

2. Nonabelian case

The nonabelian case is more complicated for the refined algebraic quantization approach [8]. The Hermitian parts of constraints $\check{\Lambda}_a$ ($\check{\Lambda}_a^+ = \check{\Lambda}_a$) satisfy the following closed-algebra relations:

$$[\check{\Lambda}_a; \check{\Lambda}_b] = i f^c_{ab} \check{\Lambda}_c. \tag{2.7}$$

for some structure constants f^c_{ab}. Let L_a, $a = \overline{1, M}$ be generators of the Lie algebra with the following commutation relations $[L_a, L_b] = i f^c_{ab} L_c$. Consider the corresponding Lie group G and the exponential mapping $\mu^a L_a \mapsto \exp(i\mu^a L_a)$. The operators $\check{\Lambda}_a$ form a representation of the Lie algebra, so that $\exp(i\mu^a \check{\Lambda}_a)$ will form a representation of group $\check{T}(\exp(i\mu^a L_a)) = \exp(i\mu^a \check{\Lambda}_a)$. By $Ad(L_a)$ we denote the adjoint representation of the Lie algebra, $(Ad(L_a)\rho)^c = i f^c_{ab} \rho^b$, while $Ad\{g\}$ is an adjoint representation of the group $(Ad\{g\}\rho)^c = (\exp(A))^c_b \rho^b$ with $A^c_b = -\mu^a f^c_{ab}$, $g = \exp(i\mu^a L_a)$.

For the general closed-algebra case, the inner product is expressed via the integral over gauge group with the help of the Giulini-Marolf group averaging formula [8] instead of (2.4):

$$\int d_R g (det Ad\{g\})^{-1/2} (\Phi, \check{T}(g)\Phi) \tag{2.8}$$

Here $d_R g$ is the right-invariant Haar measure on the group G.

Formula (2.4) is indeed a partial case of (2.8). For the abelian case considered above, one has $d_R g = d\mu^1 ... d\mu^M$, $\check{T}(g) = \exp\{i\mu^a \check{\Lambda}_a\} = \exp\{i\mu^a \hat{\Lambda}_a\}$, $\det Ad\{g\} = 1$, so that formula (2.8) takes the form

$$\int d\mu^1 ... d\mu^M (\Phi, \exp\{i\mu^a \hat{\Lambda}_a\}\Phi) \tag{2.9}$$

Integrating over μ, we obtain formula (2.4).

The case of discrete spectrum of constraints can be also considered within framework of eq.(2.8).

Example 2. Let $M = 1$, $q = \varphi \in (0, 2\pi)$, $\hat{\Lambda} = -i\frac{\partial}{\partial\varphi}$, the wave functions obey the periodic boundary conditions $\Phi(\varphi + 2\pi) = \Phi(\varphi)$. Formula (2.9) takes the form $\int_0^{2\pi} d\varphi \int d\mu \Phi^*(\varphi)\Phi(\varphi + \mu)$. If one performed the integration over $|mu \in (-\infty, +\infty)$, the inner product would be divergent. However, one should take into account that $\exp\{2\pi i \hat{\Lambda}\} = 1$. Therefore, the gauge group is $U(1)$, so that the integration in eq.(2.9) should be performed only over $\mu \in (0, 2\pi)$. For the inner product (2.9), we obtain then formula $| \int_0^{2\pi} d\varphi \Phi(\varphi)|^2$ which is a basis of the projection operator quantization [9].

Two states Φ_1 and Φ_2 are called gauge-equivalent if their difference satisfies the condition

$$\int d_R g (det Ad\{g\})^{-1/2} \check{T}(g)(\Phi_1 - \Phi_2) = 0.$$

For example, states X and $(det Ad\{h\})^{-1/2}\check{T}(h)X$ are equivalent. This means that

$$[(det Ad\{h\})^{-1/2}\check{T}(h) - 1]X \sim 0.$$

After substitution $h = \exp(i\rho^a L_a)$ we find in the leading order in ρ that

$$\hat{\Lambda}_a X \sim 0 \tag{2.10}$$

with

$$\hat{\Lambda}_a = \check{\Lambda}_a - \frac{i}{2} f^b_{ab} \tag{2.11}$$

The fact that constraints in the non-unimodular case should be not Hermitian was discussed in [8,14] in details.

The Dirac wave function can be specified as

$$\Psi = \int d_R g (det Ad\{g\})^{-1/2} T(g)\Phi. \tag{2.12}$$

analogously to eq.(2.6). It obeys the condition [8]

$$(det Ad\{h\})^{1/2}\check{T}(h)\Psi = \Psi$$

which can be also presented in the infinitesimal form

$$\tilde{\Lambda}^+_a \Psi \equiv (\Lambda_a + \frac{i}{2} f^b_{ab})\Psi = 0 \tag{2.13}$$

found in [14].

We are going to generalize the refined algebraic quantization approach to the case of nontrivial structure functions U_{ab}^c being operators rather than constants. It has been stressed in [12] that generalization of the Giulini-Marolf formula (2.8) to the open-algebra case is an interesting open problem.

One can hope that the inner product formula for the auxiliary states should be looked for in the following form

$$(\Phi, \eta\Phi) \tag{2.14}$$

for some operator η such that $\eta^+ = \eta$, $\eta \geq 0$. The main requirement for the operator η is

$$\eta\hat{\Lambda}_a = 0. \tag{2.15}$$

Two auxiliary states are called equivalent if their difference $\Delta\Phi$ has zero norm, this is certainly the case if

$$\Delta\Phi = \hat{\Lambda}_a Y^a$$

The classes of equivalence being elements of the factorspace $\mathcal{H}_{aux}/\sim$ correspond to the Dirac states with the help of formula

$$\Psi = \eta\Phi. \tag{2.16}$$

The constrained conditions (2.1) are automatically satisfied then. The space of physical states is defined as a completeness of the factorspace $\overline{\mathcal{H}_{aux}/\sim}$.

However, such a generalization is not trivial. To find it, it will be necessary to discuss a relationship between the Dirac and refined algebraic quantization approaches and BRST-BFV quantization technique.

C. BRST-BFV approach

To develop the BRST-BFV approach [1,2,4,3], it is necessary to introduce additional degrees of freedom: Lagrange multipliers and momenta λ^a, π_a, $a = \overline{1,M}$, ghosts and antighosts $C^a, \overline{C}_a$, canonically conjugated momenta $\overline{\Pi}_a, \Pi^a$, $a = \overline{1,M}$. The nontrivial (anti)commutation relations are:

$$[\lambda^a, \pi_b] = i\delta^a_b, \qquad [C^a, \overline{\Pi}_b]_+ = \delta^a_b, \qquad [\overline{C}_a, \Pi^b]_+ = \delta^b_a$$

Operators $\overline{C}_a$ and Π^b are anti-Hermitian, others are Hermitian. The main object of the BRST-BFV method is the B-charge Ω. For the closed-algebra case, it has the form

$$\Omega = C^a\check{\Lambda}_a - \frac{i}{2}f^a_{bc}\overline{\Pi}_a C^b C^c - \frac{i}{2}f^a_{ba}C^b - i\pi_a\Pi^a. \tag{2.17}$$

It is formally Hermitian and nilpotent,

$$\Omega^+ = \Omega; \qquad \Omega^2 = 0. \tag{2.18}$$

For the open-algebra case with nontrivial structure functions, the B-charge is looked for in the following form:

$$\Omega = -i\pi_a\Pi^a + C^a\hat{\Lambda}_a + ... + \Omega^{n\, b_1...b_{n-1}}_{a_1...a_n}\overline{\Pi}_{b_1}...\overline{\Pi}_{b_{n-1}}C^{a_1}...C^{a_n} + ... \tag{2.19}$$

The operators $\overline{\Pi}$ and C are ordered in formula (2.19) in such a way that ghosts C are put to the right, while the momenta $\overline{\Pi}$ are put to the left. The operator-valued coefficient functions $\Omega^{n\, b_1...b_{n-1}}_{a_1...a_n}$ being antisymmetric separately with respect to $b_1, ..., b_{n-1}$ and separately with respect to $a_1, ..., a_n$ are constructed in a standard way [3] from recursive relations that are corollaries of the properties (2.18). Formula (2.19) is in agreement with relation (2.11) since the coefficient of C^a in eq.(2.17) is indeed $\hat{\Lambda}_a$.

Instead of requirement (2.1), the BRST-BFV condition is imposed on physical states Υ:

$$\Omega\Upsilon = 0, \tag{2.20}$$

The gauge freedom is also allowed, the gauge transformation is

$$\Upsilon \to \Upsilon + \Omega X, \tag{2.21}$$

so that states Υ and $e^{[\Omega,\rho]_+}\Upsilon$ are also equivalent.

Another requirement is that physical states should be of zero ghost number,

$$N = \Pi^a \overline{C}_a - \overline{\Pi}_a C^a,$$

so that

$$N\Upsilon = 0. \tag{2.22}$$

The most nontrivial problem is to introduce an inner product in the BRST-BFV formalism. Consider the Schrodinger representation for the BFV wave function Υ, $\Upsilon = \Upsilon(q,\lambda,\Pi,\overline{\Pi})$. The operators are rewritten then as

$$C^a = \frac{\partial}{\partial \overline{\Pi}_a}; \qquad \overline{C}_a = \frac{\partial}{\partial \Pi^a}; \qquad \pi_a = -i\frac{\partial}{\partial \lambda^a}; \qquad p_i = -i\frac{\partial}{\partial q^i}, \tag{2.23}$$

the left derivatives are considered here. The inner product is indefinite. Formally, it is as follows [15]

$$(\Upsilon_1, \Upsilon_2) = \int dq \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a (\Upsilon_1(q, i\mu, \Pi, \overline{\Pi}))^* \Upsilon_2(q, -i\mu, \Pi, \overline{\Pi}). \tag{2.24}$$

The integration and conjugation rules are $(\overline{\Pi}_{a_1}...\overline{\Pi}_{a_l}\Pi^{b_1}...\Pi^{b_s})^* = (-1)^s \Pi^{b_s}...\Pi^{b_1}\overline{\Pi}_{a_l}...\overline{\Pi}_{a_1}$, $\int d\overline{\Pi}_a \overline{\Pi}_a = 1$, $\int d\Pi^a \Pi^a = 1$. However, the inner product space (2.24) requires additional investigation. For example, a class of allowed BFV wave functions Υ should be specified.

For the abelian case, one introduces the creation and annihilation operators

$$A_a^{\pm} = \frac{1}{\sqrt{2}}[\pi_a \pm i M_a{}^b \Lambda_b] \tag{2.25}$$

for some Hermitian real positively definite nondegenerate matrix M, shows [16] that it is possible to perform such a gauge transformation (2.21) that after it

$$A_a^- \Upsilon = 0. \tag{2.26}$$

Then the inner product (2.24) is shown to be convergent [16].

The more general formula for the inner product was written in [17]. First, one considers the representatives of the equivalence classes which obey the following additional conditions

$$C^a \Upsilon = 0, \qquad \pi_a \Upsilon = 0 \tag{2.27}$$

which make the state Υ BRST-BFV-invariant. Unfortunately, the quantity (Υ,Υ) is ill-defined. However, the expression

$$(\Upsilon, e^{t[\Omega,\rho]_+}\Upsilon) \tag{2.28}$$

which is formally equivalent to (Υ,Υ) occurs to be well-defined for a certain choice of the gauge fermion ρ,

$$\rho = -\lambda^a \overline{\Pi}_a. \tag{2.29}$$

Let us analyze the prescriptions for the inner products of different quantization methods and find a correspondence between states Υ, Φ, Ψ.

III. CORRESPONDENCE OF STATES

A. Abelian case

Let us investigate the inner product (2.24) under condition (2.26). Relation (2.25) and B-condition (2.20) imply that

$$[\frac{\partial}{\partial\overline{\Pi}_a} + M_b{}^a\Pi^b]\Upsilon = 0$$

so that

$$\Upsilon(q,\lambda,\Pi,\overline{\Pi}) = \exp[-\overline{\Pi}_a M_b{}^a\Pi^b]\Upsilon_0(q,\lambda).$$

Condition (2.26) implies that

$$\Upsilon_0(q,\lambda) = \exp[\lambda^a M_a{}^b\Lambda_b]\Phi(q),$$

so that

$$\Upsilon(q,\lambda,\Pi,\overline{\Pi}) = \exp[-\overline{\Pi}_a M_b{}^a\Pi^b]\exp[\lambda^a M_a{}^b\Lambda_b]\Phi(q). \tag{3.1}$$

Substituting this expression to the inner product (2.24), one finds

$$(\Upsilon,\Upsilon) = \int dq \prod_{a=1}^{M} d\mu^a \Phi^*(q) e^{-2i\mu^a M_a{}^b\Lambda_b}\Phi(q) \int \prod_{a=1}^{M} d\overline{\Pi}_a d\Pi^a \exp[-2\overline{\Pi}_a M_b{}^a\Pi^b].$$

Integration over Grassmannian variables gives us the factor $\det 2M$ which is involved to the integration measure after substitution $2\mu^a M_a{}^b = \tilde{\mu}^b$. We obtain that

$$(\Upsilon,\Upsilon) = (\Phi, \prod_{a=1}^{M} 2\pi\delta(\hat{\Lambda}_a)\Phi). \tag{3.2}$$

This formula can be valid for the case of the continuous spectrum of constraints. For the discrete spectrum case, we see that there are internal topological problems in BRST-BFV approach (cf. [18]). A possible resolution of them is to modify formula (2.24) for the BRST-BFV inner product by integration over $\mu^1,...,\mu^M$ belonging to some domain rather than over $\mu \in \mathbf{R}^M$. Formula (3.2) coincides with the inner product in the refined algebraic quantization approach (2.4), provided that Υ is taken to the gauge (2.27) and auxiliary state Φ is

$$\Phi(q) = \Upsilon(q,0,0,0). \tag{3.3}$$

Let $\tilde{\Upsilon}$ be a physical state which does not satisfy the gauge conditions (2.27). This means that

$$\tilde{\Upsilon} = \Upsilon + \Omega X, \tag{3.4}$$

while Υ obeys gauge condition (2.27). Consider the function

$$\tilde{\Phi}(q) = \tilde{\Upsilon}(q,0,0,0)$$

and investigate the relationship between Φ and $\tilde{\Phi}$. Let

$$X = X_{00}(q,\lambda) + X_{01}^a(q,\lambda)\overline{\Pi}_a + X_{10,a}(q,\lambda)\Pi^a + ...$$

Then

$$\tilde{\Phi}(q) - \Phi(q) = \hat{\Lambda}_a X_{01}^a(q,0), \tag{3.5}$$

so that $\tilde{\Phi}$ and Φ are gauge-equivalent in sense of (2.5). Thus, if $\tilde{\Upsilon}$ is an arbitrary physical state in the BRST-BFV-approach, formula (3.3) gives us a representative of class of equivalence of auxiliary states. Thus, for abelian case the correspondence between BFV and refined algebraic quantization states is found. If one used a coordinate representation with respect to ghosts and antighosts instead of momenta representation, $\Phi(q)$ would be a component of the BFV wave function with maximal number ghosts and antighosts.

To obtain a correspondence between Dirac and BFV states, consider the integral

$$\Psi(q) = \int \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a \Upsilon(q,-i\mu,\Pi,\overline{\Pi}). \tag{3.6}$$

For the gauge (2.26), Υ has the form (3.1). Substituting it to formula (3.6), we find

$$\Psi(q) = \prod_{a=1}^{M} 2\pi\delta(\hat{\Lambda}_a)\Phi(q).$$

This formula coincides with (2.6). Therefore, to find the Dirac wave function, one should take the B-state to the gauge (2.26) and use eq.(3.6). However, formula (3.6) is valid for arbitrary physical state. Namely,

$$\int \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a (\Omega X)(q, -i\mu, \Pi, \overline{\Pi}) = \int \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a (\frac{1}{i}\frac{\partial}{\partial \mu^a}\Pi^a + \frac{\partial}{\partial \overline{\Pi}_a}\hat{\Lambda}_a) X(q, -i\mu, \Pi, \overline{\Pi}) = 0.$$

We have found that for abelian case the Dirac and BRST-BFV states are related with the help of formula (3.6). In the coordinate representation with respect to ghosts, formula (3.6) can be viewed as a component of the B-function with minimal number of ghosts and antighosts.

B. Closed-algebra case

Let us generalize formulas (3.3) and (3.6) to the nonabelian case and check them for the Batalin-Marnelius prescription for the inner product.

Suppose that the B-state Υ is taken to the gauge (2.27). This means that Υ is $\lambda, \Pi, \overline{\Pi}$-independent,

$$\Upsilon = \Phi(q) \tag{3.7}$$

provided that the ghost number of Υ is zero (eq.(2.22)). Since

$$[\Omega, \rho]_+ = -\lambda^a \Lambda_a + \frac{i}{2}\lambda^a f^b_{ab} - i\lambda^a \overline{\Pi}_b C^c f^b_{ac} - \overline{\Pi}_a \Pi^a, \tag{3.8}$$

for the simplest abelian case one has

$$(e^{t[\Omega,\rho]_+}\Upsilon)(q, \lambda, \Pi, \overline{\Pi}) = e^{-t\lambda^a\Lambda_a}\Phi(q)e^{-t\overline{\Pi}_a\Pi^a}, \tag{3.9}$$

so that

$$(\Upsilon, e^{t[\Omega,\rho]_+}\Upsilon) = \int dq \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a \Phi^*(q) e^{it\mu^a\Lambda_a} e^{-t\overline{\Pi}_a\Pi^a}\Phi(q).$$

Integration over ghost variables gives us t^M, so that

$$(\Upsilon, e^{t[\Omega,\rho]_+}\Upsilon) = (\Phi, \prod_{a=1}^{M} 2\pi\delta(\hat{\Lambda}_a)\Phi). \tag{3.10}$$

We see that inner products (2.28) and (2.4) coincide, provided that correspondence between Φ and Υ is of the form (3.3). Thus, formula (3.3) is justified even if the Marnelius inner product is introduced in the abelian theory. We also see that for the case of discrete spectrum of $\hat{\Lambda}_a$ topological problems arise in this version of the BRST-BFV approach as well.

Formula (3.6) contains factors $0 \times \infty$ and should be then regularized as

$$\Psi(q) = \int \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a (e^{t[\Omega,\rho]_+}\Upsilon)(q, -i\mu, \Pi, \overline{\Pi}).$$

Evaluating this integral, we also obtain formula (2.6). Formula (3.6) is justified.

Consider now the nonabelian case. Let us look for the wave function $e^{t[\Omega,\rho]_+}\Upsilon$ in the following form:

$$(e^{t[\Omega,\rho]_+}\Upsilon)(q, \lambda, \Pi, \overline{\Pi}) = e^{-t\lambda^a\hat{\Lambda}_a}\Phi(q)e^{\overline{\Pi}_a B^a{}_b(\lambda,t)\Pi^b}$$

where $\hat{\Lambda}_a$ is of the form (2.11) Making use of the relation $\frac{d}{dt}(e^{t[\Omega,\rho]_+}\Upsilon) = [\Omega, \rho]_+ e^{t[\Omega,\rho]_+}\Upsilon$, we find the following equation for the matrix B,

$$\dot{B}^b{}_d = -i\lambda^a f^b_{ac} B^c{}_d - 1,$$

so that

$$B(\lambda, t) = -\int_0^t d\tau Ad\{\exp(-\tau\lambda^a L_a\}\}.$$

Therefore,

$$(\Upsilon, e^{t[\Omega,\rho]_+}\Upsilon) = \int dq \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a \Phi^*(q) e^{it\mu^a(\check{\Lambda}_a - \frac{i}{2}f^b_{ab})}\Phi(q) e^{-\overline{\Pi}_a \int_0^t d\tau (Ad\{\exp(i\tau\mu^c L_c)\})^a_b \Pi^b}. \quad (3.11)$$

Integration over fermionic variables gives us the group measure

$$dg = \det \int_0^t d\tau (Ad\{\exp(i\tau\mu^c L_c)\}) \prod_{a=1}^M d\mu^a, \qquad g = \exp(it\mu^c L_c)$$

It happens that it coincides with the right-invariant Haar measure which has the form (see, for example, [19]) $d_R g = d\mu J(\mu)$ with $J(\mu) = \det \frac{\delta\rho}{\delta\mu}$ for

$$\exp(i(\mu^a + \delta\mu^a)L_a) = \exp(i\delta\rho^a L_a)\exp(i\mu^a L_a).$$

Without loss of generality, consider the case $t = 1$. One finds

$$\delta\rho^a L_a = \int_0^1 d\alpha e^{i\alpha\mu^a L_a}\delta\mu^b L_b e^{-i\alpha\mu^a L_a} = \int_0^1 d\alpha (Ad\{e^{i\alpha\mu^c L_c}\}\delta\mu)^a L_a, \quad (3.12)$$

so that $dg = d_R g$. The multiplicator $e^{t\frac{1}{2}\mu^a f^b_{ab}}$ can be presented as $(detAd\{g\})^{-1/2}$. The inner product (3.11) occurs to coincide with (2.8), provided that the correspondence between Φ and Υ is of the form (2.17). Under gauge transformation (3.4) of Υ the auxiliary state Φ is transformed according to eq.(3.5). This is a gauge transformation (2.5). Thus, formula (3.3) is checked for the nonabelian case as well.

Analogously to the abelian case, one can propose that each point of the gauge group should be taken into account once. This means that the inner product (2.24) should be in general modified: integration over μ should be performed over some domain only.

The Dirac wave function (2.12) can be also presented via the integral over ghost momenta and Lagrange multipliers,

$$\Psi(q) = \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a (e^{t[\Omega;\rho]_+}\Phi)(q, -i\mu, \Pi, \overline{\Pi}). \quad (3.13)$$

Since states Φ and $e^{t[\Omega;\rho]_+}\Phi$ are formally BFV-equivalent, one can notice that eq.(3.13) is formally equivalent to (3.6). Namely, gauge-equivalent BFV-states give us identical Dirac wave functions, since the BRST-BFV charge can be written as a full derivative,

$$\Omega = (\check{\Lambda}_a + \frac{i}{2}f^b_{ab})\frac{\partial}{\partial\overline{\Pi}_a} - \frac{i}{2}f^a_{bc}\frac{\partial}{\partial\overline{\Pi}_b}\frac{\partial}{\partial\overline{\Pi}_c}\overline{\Pi}_a - \frac{\partial}{\partial\lambda^a}\Pi^a,$$

so that the integral of ΩX over μ, $\overline{\Pi}$, Π vanishes. Furthermore, it follows from the property

$$\int \prod_a d\mu^a d\overline{\Pi}_a d\Pi^a (\Omega\overline{\Pi}_a \Upsilon)(q, -i\mu, \Pi, \overline{\Pi})$$

and relation $\Omega\Upsilon = 0$ that eq.(2.13) is indeed satisfied for definition (3.6).

Thus, the formal relationship between Dirac and BFV states is obtained. We also see that integration over μ should be performed carefully due to topological problems.

IV. A PROPOSAL FOR THE INNER PRODUCT AND ITS PROPERTIES FOR THE OPEN-ALGEBRA CASE

A. Prescription for the inner product

To develop the method of refined algebraic quantization for the case of nontrivial structure functions, let us suppose that formula (3.3) for the correspondence between BFV and refined algebraic states is valid. Then we will write down the Batalin-Marnelius prescription for BFV inner product and find the operator η entering to formula (2.14).

The quantum constrained system is specified by the B-charge (2.19). Let $\Upsilon = \Phi(q)$ satisfy conditions (2.27). Calculate the inner product (2.28). One has

$$[\Omega,\rho]_+ = -\overline{\Pi}_a \Pi^a - \lambda^a \hat{\Omega}_a$$

with

$$\hat{\Omega}_a = \Omega_a(\overline{\Pi}, C) = [\overline{\Pi}_a, \Omega]_+ = \hat{\Lambda}_a + ... + n\Omega^{n b_1 ... b_{n-1}}_{a_1 ... a_{n-1} a} \overline{\Pi}_{b_1} ... \overline{\Pi}_{b_{n-1}} C^{a_1} ... C^{a_{n-1}} + ...$$

We obtain then that

$$(\Phi, e^{t[\Omega,\rho]_+}\Phi) = \int dq \Phi^*(q) \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a e^{-t\overline{\Pi}_a \Pi^a + it\mu_a \hat{\Omega}_a} \Phi(q) \tag{4.1}$$

with $\hat{\Omega}_a = \Omega_a(\overline{\Pi}, \partial/\partial\overline{\Pi})$. Formula (4.1) is of the type (2.14) with

$$\eta = \int \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a e^{-\overline{\Pi}_a \Pi^a + i\mu_a \hat{\Omega}_a(\overline{\Pi}, \partial/\partial\overline{\Pi})} 1. \tag{4.2}$$

Here Π^a and μ_a are rescaled in t times.

One should take into account the topological problems analogously to the closed-algebra case: integration may be performed not over all values of μ but over μ belonging to some domain.

B. Properties of the inner product

Let us investigate properties of the operator η (4.2). First of all, check that $\eta^+ = \eta$, so that formula (2.14) gives us real values. One has

$$(\Phi, \eta\Phi) = \int \prod_{a=1}^{M} d\mu^a (\Phi, \exp[\Pi^a \overline{\Pi}_a + i\mu_a \hat{\Omega}_a]\Phi)$$

and

$$(\Phi, \eta\Phi)^* = \int \prod_{a=1}^{M} d\mu^a (\Phi, \exp[\Pi^a \overline{\Pi}_a - i\mu_a \hat{\Omega}_a^+]\Phi)$$

After change of variables $\mu_a \to -\mu_a$ and using the property $\hat{\Omega}_a^+ = \hat{\Omega}_a$ being a corollary of the relations $\overline{\Pi}_a^+ = \overline{\Pi}_a$ and $\Omega^+ = \Omega$, we find $\eta^+ = \eta$.

Let us check relation (2.15). One has

$$\eta \hat{\Lambda}_b Y^b(q) = \int \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a \exp[\Pi^a \overline{\Pi}_a + i\mu_a \hat{\Omega}_a] \Omega \overline{\Pi}_b Y^b(q). \tag{4.3}$$

Since $\Omega^2 = \Omega$, the operators Ω and $[\Omega,\rho]_+$ commute:

$$\Omega[\Omega,\rho]_+ = \Omega\rho\Omega = [\Omega,\rho]_+ \Omega, \tag{4.4}$$

so that

$$e^{\Pi^a\overline{\Pi}_a+i\mu_a\hat{\Omega}_a}\Omega=\Omega e^{\Pi^a\overline{\Pi}_a+i\mu_a\hat{\Omega}_a}.$$

Formula (4.3) transforms then to

$$\eta\hat{\Lambda}_bY^b(q)=\int\prod_{a=1}^{M}d\mu^a d\overline{\Pi}_a d\Pi^a\Omega^+\exp[\Pi^a\overline{\Pi}_a+i\mu_a\hat{\Omega}_a]\overline{\Pi}_bY^b(q) \tag{4.5}$$

since $\Omega=\Omega^+$. The operator Ω^+ can be presented in representation (2.23) as

$$\Omega^+=\frac{1}{i}\frac{\partial}{\partial\mu_a}\Pi^a+\frac{\partial}{\partial\overline{\Pi}_a}\hat{\Lambda}_a^+ +...+(\Omega^{n\,b_1...b_{n-1}}_{a_1...a_n})^+\frac{\partial}{\partial\overline{\Pi}_{a_n}}...\frac{\partial}{\partial\overline{\Pi}_{a_1}}\overline{\Pi}_{b_{n-1}}...\overline{\Pi}_{b_1}+... \tag{4.6}$$

Integral (4.5) then vanishes as an integral of full derivative. Formula (2.15) is checked. Thus, formula (4.2) obeys the desired properties of the operator η entering to the inner product. However, the problem of positive definiteness of the inner product remains to be investigated.

C. Correspondence of states for the case of nontrivial structure functions

Let us show that correspondence of BFV, auxiliary and Dirac states given by eqs.(3.3) and (3.6) remains valid for the case of nontrivial structure functions.

First, notice that the auxiliary state $(\Omega X)(q,0,0,0)$ is of the form

$$(\Omega X)(q,0,0,0)=\hat{\Lambda}_a\frac{\partial}{\partial\overline{\Pi}_a}|_{\overline{\Pi}=\Pi=0,\lambda=0}X$$

and is equivalent then to zero. Thus, refined algebraic quantization state $\Upsilon(q,0,0,0)=(\Omega X)(q,0,0,0)+\Phi(q)$ is equivalent to $\Phi(q)$. Formula (3.3) is justified.

The Dirac wave function (2.16) can be presented as

$$\Psi(q)=\int\prod_{a=1}^{M}d\mu^a d\overline{\Pi}_a d\Pi^a(e^{[\Omega,\rho]_+}\Phi)(q,-i\mu,\Pi,\overline{\Pi}).$$

To check formula (3.6), it is sufficient to justify that equivalent BFV states (2.21) give equal Dirac wave functions (3.6). However, it follows directly from (4.6) that

$$\int\prod_{a=1}^{M}d\mu^a d\overline{\Pi}_a d\Pi^a(\Omega^+X)(q,-i\mu,\Pi,\overline{\Pi})=0$$

since integrals of full derivatives vanish. Formula (3.6) is obtained.

To check relation (2.1), use the property

$$\int\prod_{a=1}^{M}d\mu^a d\overline{\Pi}_a d\Pi^a(\Omega^+\overline{\Pi}_a\Upsilon)(q,-i\mu,\Pi,\overline{\Pi})=0.$$

Since $\Omega^+\Upsilon=\Omega\Upsilon=0$, one can rewrite it as follows,

$$\int\prod_{a=1}^{M}d\mu^a d\overline{\Pi}_a d\Pi^a([\Omega^+,\overline{\Pi}_a]_+\Upsilon)(q,-i\mu,\Pi,\overline{\Pi})=0. \tag{4.7}$$

The anticommutator has the form

$$[\Omega^+;\overline{\Pi}_a]=\hat{\Lambda}_a^+ +...+n(\Omega^{n\,b_1...b_{n-1}}_{a_1...a_{n-1}a})^+\frac{\partial}{\partial\overline{\Pi}_{a_{n-1}}}...\frac{\partial}{\partial\overline{\Pi}_{a_1}}\overline{\Pi}_{b_{n-1}}...\overline{\Pi}_{b_1}+...$$

It contains full derivatives, except for the term $\hat{\Lambda}_a^+$. Thus, eq.(4.7) can be presented as

$$\hat{\Lambda}_a^+ \int \prod_{a=1}^{M} d\mu^a d\overline{\Pi}_a d\Pi^a \Upsilon(q, -i\mu, \Pi, \overline{\Pi}) = 0.$$

Eq.(2.1) is obtained.

Thus, the *formal* relationship between refined algebraic quantization, Dirac and BFV states is found. However, there are topology problems which may lead to integration over some domain in (3.6). They are to be investigated in future.

V. CORRESPONDENCE OF OBSERVABLES

Let us consider the properties of quantum observables in different quantization approaches.

In the BRST-BFV approach, observables are viewed as series

$$H_B = H + ... + H^{n}{}^{b_1...b_n}_{a_1...a_n} \overline{\Pi}_{b_1}...\overline{\Pi}_{b_n} C^{a_1}...C^{a_n} + ... \tag{5.1}$$

The operator coefficient functions $H^{n}{}^{b_1...b_n}_{a_1...a_n}(\hat{p}, \hat{q})$ are chosen in such a way that

$$H_B^+ = H_B, \qquad [\Omega, H_B] = 0. \tag{5.2}$$

These properties provide that physical states (2.20) are taken by the operator H to physical, while equivalent states are taken to equivalent; the inner product is conserved under evolution.

One has:

$$\Omega H_B = C^c \hat{\Lambda}_c H + \hat{\Lambda}_b H_a^{1b} C^a + ..., \qquad H_B \Omega = H C^c \hat{\Lambda}_c + ...$$

where ... are terms with ghost momenta. Therefore, H should obey the following property:

$$[H; \hat{\Lambda}_a] = \hat{\Lambda}_b H_a^{1b}$$

for some operators H_a^{1b}. We have obtained relation (2.3).

Since $(H_B \Upsilon)(q, 0, 0, 0) = H\Upsilon(q, 0, 0, 0)$, it is the operator H that corresponds to the B-observable (5.1) in the refined algebraic quantization approach. An important feature of the physical observable is that the corresponding evolution operator e^{-iHt} should be unitary with respect to the inner product (2.14). This means that

$$(e^{-iHt})^+ \eta e^{-iHt} = \eta.$$

or

$$H^+ \eta = \eta H. \tag{5.3}$$

This property is to be checked.

Let Φ be an auxiliary state corresponding to the Dirac state $\Psi = \eta\Phi$. The observable H takes it to $H\Phi$. This corresponds to the Dirac state

$$\eta H \Phi = H^+ \eta \Phi = H^+ \Psi.$$

Therefore, it is the operator H^+ that corresponds to the observable H in the Dirac approach, while $\exp(-iH^+t)$ is an evolution operator. This is also a corollary of (3.6) since

$$\int \prod_{a=1}^{M} d\mu_a d\overline{\Pi}_a d\Pi^a H_B^+ \Upsilon(q, -i\mu, \Pi, \overline{\Pi}) = H^+ \int \prod_{a=1}^{M} d\mu_a d\overline{\Pi}_a d\Pi^a \Upsilon(q, -i\mu, \Pi, \overline{\Pi})$$

because integral of full derivative vanishes.

A. "Closed-algebra" case

Let us check formula (5.3). First of all, consider the "closed-algebra" case with constant operators $H_a^{1b} = iR_a^b$, $R_a^b = const$. The higher-order terms of expression (5.1) vanish then [3], so that

$$H_B = H + i\overline{\Pi}_b R_c^b C^c. \tag{5.4}$$

Eq.(5.3) to be checked can be rewritten as

$$\int d_R g H^+ e^{i\mu^a \hat{\Lambda}_a} = \int d_R g e^{i\mu^a \hat{\Lambda}_a} H. \tag{5.5}$$

One has

$$e^{i\mu^a\hat{\Lambda}_a} H e^{-i\mu^a\hat{\Lambda}_a} = H + \int_0^1 d\alpha e^{i\alpha\mu^a\hat{\Lambda}_a}[i\mu^a\hat{\Lambda}_a; H]e^{-i\mu^a\hat{\Lambda}_a} = H + \int_0^1 d\alpha e^{i\alpha\mu^a\hat{\Lambda}_a}\mu^a R_a^b \hat{\Lambda}_b e^{-i\mu^a\hat{\Lambda}_a}.$$

Since the commutation relations between generators $\hat{\Lambda}_a$ coincide with (2.7), $[\hat{\Lambda}_a, \hat{\Lambda}_b] = if_{ab}^c\hat{\Lambda}_c$, it follows from eq.(3.12) that

$$e^{i\mu^a\hat{\Lambda}_a} H e^{-i\mu^a\hat{\Lambda}_a} = H + \frac{1}{i}\frac{d}{d\tau}|_{\tau=0} e^{i(\mu^a+\tau\mu^b R_b^a)\hat{\Lambda}_a} e^{-i\mu^a\hat{\Lambda}_a}.$$

Eq. (5.5) is taken then to the form $\int d_R g (H^+ - H) e^{i\mu^a\hat{\Lambda}_a} = \int d_R g \frac{1}{i}\frac{d}{d\tau}|_{\tau=0} e^{i(\mu^a+\tau\mu^b R_b^a)\hat{\Lambda}_a}$, so that

$$H = H^+ - iR_b^b \tag{5.6}$$

Condition (5.6) is a relationship between observables H and H^+ in the projection operator and Dirac approaches. We see that this is in agreement with the condition $H_B^+ = H_B$.

B. General case

Let us verify formula (5.3) for general case. One has

$$\eta H \Phi(q) = \int \prod_{a=1}^M d\mu_a d\overline{\Pi}_a d\Pi^a \exp[\Pi^a \overline{\Pi}_a + i\mu_a \hat{\Omega}_a] H_B \Phi(q), \tag{5.7}$$

while

$$H^+ \eta \Phi(q) = \int \prod_{a=1}^M d\mu_a d\overline{\Pi}_a d\Pi^a H_B^+ \exp[\Pi^a \overline{\Pi}_a + i\mu_a \hat{\Omega}_a] \Phi(q). \tag{5.8}$$

Here we have taken into account that $C^a\Phi(q) = 0$ and that the integral of full derivative vanishes. Consider the difference of eqs.(5.7), (5.8). Let us make use of the following relation,

$$H_B^+ e^{[\Omega,\rho]_+} - e^{[\Omega,\rho]_+} H_B = \int_0^1 d\tau e^{\tau[\Omega,\rho]_+}[[\Omega,\rho]_+, H_B] e^{(1-\tau)[\Omega,\rho]_+},$$

since $H_B^+ = H_B$. Moreover, $[[\Omega,\rho]_+, H_B] = [\Omega, [H_B, \rho]]_+$. It follows from eq.(4.4) that

$$H_B^+ e^{[\Omega;\rho]_+} - e^{[\Omega;\rho]_+} H_B = [\Omega; A]_+$$

with

$$A = \int_0^1 d\tau e^{\tau[\Omega,\rho]_+}[H_B; \rho] e^{(1-\tau)[\Omega,\rho]_+},$$

Therefore, the difference between formulas (5.7) and (5.8) reads

$$(H^+\eta - \eta H)\Phi(q) = \int \prod_{a=1}^M d\mu^a d\overline{\Pi}_a d\Pi^a [\Omega^+ A + A\Omega] \Phi(q).$$

This integral vanishes since $\Omega\Phi(q) = 0$ and an integral of full derivative is zero. Thus, relation (5.3) is satisfied.

VI. A SIMPLE EXAMPLE OF SYSTEM WITH NONTRIVIAL STRUCTURE FUNCTIONS

Consider a simple example of a system with structure functions. Investigate the model with 3 degrees of freedom (p_i, q^i), $i = \overline{1,3}$ and 2 classical constraints

$$\Lambda_1 = a(q^2, q^3)p_1, \qquad \Lambda_2 = p_2. \tag{6.1}$$

Since $\{\Lambda_1, \Lambda_2\} = \partial_2 \log a(q^2, q^3)\Lambda_1$, the constraints forms an algebra with structure functions. Let us look for the B-charge in the form (2.19). In classical theory, it should be written as

$$\Omega = -i\pi_1\Pi^1 - i\pi_2\Pi^2 + p_1aC^1 + p_2C^2 + (\alpha_1\overline{\Pi}_1 + \alpha_2\overline{\Pi}_2)C^1C^2 \tag{6.2}$$

for some functions $\alpha_a(p, q)$. The property $\{\Omega, \Omega\} = 0$ means that

$$p_1a\alpha_1 + p_2\alpha_2 = [p_1a; p_2],$$

so that

$$\alpha_1 = -i\partial_2 \log a; \qquad \alpha_2 = 0.$$

We see that classically

$$\Omega = -i\pi_1\Pi^1 - i\pi_2\Pi^2 + p_1aC^1 + p_2C^2 - i\partial_2 \log a\overline{\Pi}_1C^1C^2.$$

To quantize the B-charge, one should choose the operator ordering. If the $\overline{\Pi}$-operators were put to the left with respect to C-operators, the quantum B-charge would be not Hermitian. To obey the condition $\Omega^+ = \Omega$, let us use the Weyl quantization

$$\Omega = -i\pi_1\Pi^1 - i\pi_2\Pi^2 + p_1aC^1 + (p_2 - i\overline{\Pi}_1\partial_2 \log aC^1 + \frac{i}{2}\partial_2 \log a)C^2 \tag{6.3}$$

It is remarkable that in quantum theory the constraint $\hat{\Lambda}_2$ should be modified with respect to the classical theory (6.1); it follows from eq.(2.19) that

$$\hat{\Lambda}_2 = p_2 + \frac{i}{2}\partial_2 \log a,$$

so that the operator $\hat{\Lambda}_2$ becomes formally non-Hermitian. This feature of quantum constraints is known from the theory of constrained systems with nonunimodular closed algebra [14,8].

Let us evaluate the inner product (4.1). Consider the wave function

$$\Upsilon^t(q, \overline{\Pi}, \Pi) = e^{-t\overline{\Pi}_a\Pi^a + it\mu_a\hat{\Omega}_a}\Phi(q). \tag{6.4}$$

Since

$$\hat{\Omega}_1 = p_1a + i\overline{\Pi}_1\partial_2 \log aC^2, \qquad \hat{\Omega}_2 = p_2 - i\overline{\Pi}_1\partial_2 \log aC^1 + \frac{i}{2}\partial_2 \log a,$$

the state (6.4) obeys the following Cauchy problem

$$\frac{\partial}{\partial t}\Upsilon^t = [-\overline{\Pi}_1\Pi^1 - \overline{\Pi}_2\Pi^2 + a\mu_1\partial_1 + \mu_2\partial_2 - \frac{\mu_2}{2}\partial_2 \log a - \mu_1\overline{\Pi}_1\partial_2 \log a\frac{\partial}{\partial\overline{\Pi}_2} + \mu_2\overline{\Pi}_1\partial_2 \log a\frac{\partial}{\partial\overline{\Pi}_1}]\Upsilon^t, \tag{6.5}$$

$$\Upsilon^0 = \Phi(q)$$

Since eq.(6.5) is a first-order partial differential equation, it can be solved by the characteristic method. The solution is looked for in the following form

$$\Upsilon^t(Q^t, \tilde{\Pi}^t, \Pi) = \exp[\int_0^t d\tau[-\tilde{\Pi}_1^\tau\Pi^1 - \tilde{\Pi}_2^\tau\Pi^2 - \frac{\mu_2}{2}\partial_2 \log a(Q^\tau)]]\Upsilon^0(Q^0, \tilde{\Pi}^0, \Pi),$$

where the functions Q^t, $\tilde{\Pi}^t$ satisfy the following ordinary differential equations,

$$\dot{Q}_1^t = -a(Q_2, Q_3)\mu_1; \qquad \dot{Q}_2^t = -\mu_2, \qquad \dot{Q}_3^t = 0,$$

$$\frac{d}{dt}\tilde{\Pi}_2^t = \mu_1 \tilde{\Pi}_1 \partial_2 \log a(Q_2, Q_3), \qquad \frac{d}{dt}\tilde{\Pi}_1^t = -\mu_2 \tilde{\Pi}_1 \partial_2 \log a(Q_2, Q_3),$$

so that the classical characteristic trajectory is

$$Q_3^t = Q_3^0, \qquad Q_2^t = Q_2^0 - \mu_2 t, \qquad Q_1^t = Q_1^0 - \int_0^t d\tau a(Q_2^0 - \mu_2\tau, Q_3^0)\mu_1,$$

$$\tilde{\Pi}_1^t = \frac{a(Q_2^0 - \mu_2 t, Q_3^0)}{a(Q_2^0, Q_3^0)}\tilde{\Pi}_1^0, \qquad \tilde{\Pi}_2^t = \tilde{\Pi}_2^0 + \frac{1}{a(Q_2^0, Q_3^0)}\mu_1 \tilde{\Pi}_1^0 \int_0^t d\tau \partial_2 a(Q_2^0 - \mu_2\tau, Q_3^0).$$

Combining all factors, one finds the solution the Cauchy problem (6.5),

$$\Upsilon^t(x, \overline{\Pi}, \Pi) = \sqrt{\frac{a(x_2, x_3)}{a(x_2 + \mu_2 t, x_3)}} \exp[-\int_0^t d\tau \frac{a(x_2 + \mu_2 t, x_3)}{a(x_2, x_3)}\overline{\Pi}_1 \Pi^1 - t\overline{\Pi}_2 \Pi^2]$$

$$\times \exp[\int_0^t d\tau \tau \mu_1 \frac{\partial_2 \log a(x_2 + \mu_2\tau)}{a(x_2, x_3)}\overline{\Pi}_1 \Pi^2]\Phi(x_1 + \int_0^t d\tau a(x_2 + \mu_2\tau, x_3)\mu_1, x_2 + \mu_2 t, x_3) \tag{6.6}$$

One can also check by the direct computations that expression (6.6) really satisfies the Cauchy problem (6.5). The inner product (4.1) reads

$$\int dx \Phi^*(x) \prod_{a=1}^{M} d\mu_a d\overline{\Pi}_a d\Pi^a \Upsilon^t(x, \overline{\Pi}, \Pi). \tag{6.7}$$

Integration over ghost variables gives us the multiplier

$$t \int_0^t d\tau a(x_2 + \mu_2\tau, x_3)\frac{1}{a(x_2, x_3)}.$$

After rescaling of variables μ

$$\xi_1 = \int_0^t d\tau a(x_2 + \mu_2\tau, x_3)\mu_1, \qquad \xi_2 = t\mu_2$$

one finds that the integral (6.7) takes a simple form

$$\int dx_1 dx_2 dx_3 d\xi_1 d\xi_2 \Phi^*(x_1, x_2, x_3)\frac{1}{\sqrt{a(x_2 + \xi_2, x_3)a(x_2, x_3)}}\Phi(x_1 + \xi_1, x_2 + \xi_2, x_3)$$

We see that the bilinear form η can be defined as

$$(\Phi, \eta\Phi) = \int dx_3 |\int dx_1 dx_2 \frac{\Phi(x_1, x_2, x_3)}{\sqrt{a(x_2, x_3)}}|^2,$$

so that the correspondence between the Dirac wave function Ψ and the auxiliary state Φ is

$$\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{a(x_2, x_3)}}\int dy_1 dy_2 \frac{\Phi(y_1, y_2, x_3)}{\sqrt{a(y_2, x_3)}}.$$

It obeys the constraints

$$a(x_2, x_3)p_1\Psi \equiv \hat{\Lambda}_1^+\Psi = 0, \qquad \frac{1}{\sqrt{a(x_2, x_3)}}p_2\sqrt{a(x_2, x_3)}\Psi \equiv \hat{\Lambda}_2^+\Psi = 0.$$

while the gauge transformation of Φ is

$$\Phi \to \Phi + \sqrt{a(x_2, x_3)}p_2\frac{1}{\sqrt{a(x_2, x_3)}}Y^2 + a(x_2, x_3)p_1 Y^1$$

for some functions Y^1 and Y^2. We see that all properties of η (including positive definiteness) are indeed satisfied in this example.

VII. DISCUSSION

Thus, the correspondence between states and observables in BRST-BFV, Dirac and refined algebraic quantizations is found. For different versions of BRST-BFV approach, the relations (3.3) and (3.6) for auxiliary and Dirac state vectors are found. However, it has been noticed that there are internal problems in the BRST-BFV formalism. They arise when the constraint gauge group is topologically nontrivial. For simple examples, the topological problems of B-approach can be resolved by integrating over μ belonging to some domain in the inner product formula (2.24). One can hope that this prescription will work in general case as well.

The relationship between observables H_B, H and H^+ in BFV, refined algebraic and Dirac approaches is found.

Starting from BRST-BFV formula for the inner product and obtained relationship between BRST-BFV and auxiliary states, we have found expression (2.14) for the inner product of auxiliary states. The operator η is written for general case of nontrivial structure functions (eq.(4.2)). Thus, the refined algebraic quantization approach is generalized to the open-algebra case.

For a simple exactly solvable example, an explicit formula for the inner product is obtained.

A wide class of such examples of systems with structure functions can be constructed as follows. Consider the Lie-algebra constrained system: $\hat{\Lambda}_a = L_a - \frac{i}{2} f^c_{ac}$, $U^a_{bc} = f^a_{bc} = const$ such that L_a are linear in momenta, $L_a = \alpha_{aj}(x)p_j + \beta_a(x)$. The B-charge is

$$\Omega_0 = C^a L_a - \frac{i}{2} f^a_{bc} \overline{\Pi}_a C^b C^c - \frac{i}{2} f^a_{ba} C^b - i\pi_a \Pi^a.$$

Consider the unitary transformation being an exponent of the operator quadratic with respect to ghost variables,

$$U = \exp[\overline{\Pi}_a A^a_b(x) C^b - \frac{1}{2} A^a_a(x)]$$

It generates a linear canonical transformation of ghosts. The transformed B-charge $U^{-1}\Omega_0 U = \Omega$ is Hermitian and nilpotent. It contains terms Ω^1 and Ω^2 only and corresponds to the new system with classical constraints

$$\Lambda_{a'} = L_a (\exp A)^a_{a'}. \tag{7.1}$$

with quantum corrections. Generally, they form an algebra with nontrivial structure functions. Since $\Omega^n = 0$, $n \geq 3$, while the constraints are linear in momenta, the Cauchy problem analogous to (6.5) still corresponds to the first-order partial differential equation and can be solved exactly, so that it is also possible to perform an integration in eq.(4.2) explicitly.

We see that the system with classical constraints (7.1) which was mentioned in [12] can be exactly investigated by the approach proposed in this paper.

The case of an open gauge algebra corresponding to nontrivial coefficient functions Ω^n, $n \geq 3$ is much more complicated for the exact calculations. However, the integral formula (4.2) is still valid, so that one can use it for numerical calculations or for application of asymptotic methods such as perturbation theory or semiclassical approximation [20].

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